Approximate Likelihood Ratio Tests

Definition 1.1 *For testing* $H_0 : \theta \in \omega$ *versus* $H_1 : \theta \in \Theta - \omega$ *when* $X_i \stackrel{iid}{\sim} f_X(x, \theta)$*, a test which rejects for small values of*

$$\Lambda^* = \frac{\max_{\theta \in \omega}(L(\theta))}{\max_{\theta \in \Theta - \omega}(L(\theta))} \tag{1}$$

is a **likelihood ratio test.**

Rejecting for small Λ^* is equivalent to rejecting for

$$\Lambda = \frac{\max_{\theta \in \omega}(L(\theta))}{\max_{\theta \in \Theta}(L(\theta))} \tag{2}$$

Theorem 1.1 *Assuming that the density/mass function is "smooth," for*

$$\Lambda = \frac{\max_{\theta \in \omega}(L(\theta))}{\max_{\theta \in \Theta}(L(\theta))} \tag{3}$$

we have

$$-2\ln(\Lambda) \xrightarrow{d} \chi^2_{n - \dim(\theta_0)} \tag{4}$$

This theorem is useful when the distribution of $T(\mathbf{X})$ is unknown.

In the following example, we can use a likelihood ratio test without approximation because the distribution of the test statistic is known.

Example 1.1 *Let* $X \overset{iid}{\sim} B(100, p)$*. Test the hypotheses*

$$H_0 : p = 0.5 \textit{ versus } H_1 : p > 0.5$$

Now

$$\begin{aligned} L_0(p) &= \binom{n}{x}\left(\frac{1}{2}\right)^x\left(1-\frac{1}{2}\right)^{n-x} && (5)\\ &= \binom{n}{x}\left(\frac{1}{2}\right)^n && (6)\end{aligned}$$

and

$$L_1(p) = \binom{n}{x}(p)^x(1-p)^{n-x} \quad (7)$$

so that

$$\begin{aligned} l_1(p) &= \ln\binom{n}{x} + x\ln(p) + (n-x)\ln(1-p) && (8)\\ & && (9)\end{aligned}$$

and

$$\begin{aligned} \dot{l}_1(p) &= \frac{x}{p} - \frac{n-x}{1-p} && (10)\\ \frac{x}{\hat{p}} &= \frac{n-x}{1-\hat{p}} && (11)\\ x - \hat{p}x &= n\hat{p} - \hat{p}x && (12)\\ \hat{p} &= \frac{x}{n} && (13)\end{aligned}$$

We can now compute

$$\begin{aligned} \Lambda &= \frac{\max(L_0(\theta))}{\max(L_1(\theta))} && (14)\\ &= \frac{\binom{n}{x}\left(\frac{1}{2}\right)^n}{\binom{n}{x}\left(\frac{x}{n}\right)^x\left(1-\frac{x}{n}\right)^{n-x}} && (15)\\ &= \frac{1}{2^n\left(\frac{x}{n}\right)^x\left(1-\frac{x}{n}\right)^{n-x}} && (16)\end{aligned}$$

Note that Λ *is small for* $\frac{X}{n}$ *or* X *"large." So, we choose* k_α *such that* $\mathrm{P}(X \geq k_\alpha | p = 0.5) = \alpha$*.*

A little work in R can help here.

```
> x = 0:100
> plot(x,1/(2^100*(x/100)^100*(1-x/100)^(100-x)), type="l")
> qbinom(c(0.9,0.95,0.99), 100, 0.5)
  56 58 62
> cbind(55:63, pbinom(55:63,100,0.05))
  56  .9033
  58  .9557
  62  .9940
```

The following example shows how we can use approximate likelihood ratio tests when the distribution of the test statistic is unknown. The example is the basis for testing for the Poisson distribution of pottery sherds on top of Tel el-Farah South in Israel.

Example 1.2 *If we assume that pottery sherds reach the surface of the tel in a random fashion, and that the arrival of each sherd is Bernoulli, then the number of sherds per unit circle should be Poisson. We can now determine if the distribution of sherds is constant across the top of the tel.*

Consider testing the hypotheses

$$H_0 : X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} \text{Poisson}(\lambda)$$

versus

$$H_1 : X_1, X_2, \ldots, X_n \overset{\text{ind}}{\sim} \text{Poisson}(\lambda_i) \text{ for } i = 1, 2, \ldots, n$$

Under H_0 we have

$$L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \tag{17}$$

$$= \frac{\lambda^{\sum x_i} e^{-n\lambda}}{\prod(x_i!)} \tag{18}$$

so that

$$l(\lambda) = \sum x_i \ln(\lambda) - n\lambda - \sum \ln(x_i!) \tag{19}$$

and

$$\dot{l}(\lambda) = \frac{\sum x_i}{\lambda} - n \tag{20}$$

$$\frac{\sum x_i}{\widehat{\lambda}} = n \tag{21}$$

$$\widehat{\lambda} = \overline{X} \tag{22}$$

Under H_1 we find that

$$
\begin{aligned}
L(\lambda) &= \prod_{i=1}^{n} \frac{\lambda_i^{x_i} e^{-\lambda_i}}{x_i!} && (23)\\
&= \frac{\left(\prod \lambda_i^{x_i}\right) e^{-\sum \lambda_i}}{\prod (x_i!)} && (24)
\end{aligned}
$$

so that

$$
l(\lambda) = \sum x_i \ln(\lambda_i) - \sum \lambda_i - \sum \ln(x_i!) \qquad (25)
$$

Partial derivatives lead to

$$
\widetilde{\lambda}_i = x_i \qquad (26)
$$

We now compute Λ using the maximum likelihood results for both the numerator and denominator.

$$
\begin{aligned}
\Lambda &= \frac{\prod_{i=1}^{n} \left(\widehat{\lambda}^{x_i} e^{-\widehat{\lambda}}/(x_i!)\right)}{\prod_{i=1}^{n} \left(\widetilde{\lambda}_i^{\,x_i} e^{-\widetilde{\lambda}_i}/\,(x_i!)\right)} && (27)\\
&= \prod \left(\frac{\overline{x}}{x_i}\right)^{x_i} e^{x_i - \overline{x}} && (28)
\end{aligned}
$$

The distribution of this statistic is unknown so we turn to asymptotics.

$$
\begin{aligned}
-2\ln(\Lambda) &= -2 \sum \left[X_i \ln\left(\frac{\overline{X}}{X_i}\right) + (X_i - \overline{X}) \right] && (29)\\
&= 2 \sum X_i \ln\left(\frac{X_i}{\overline{X}}\right) + \sum X_i - n\overline{X} && (30)\\
&= 2 \sum X_i \ln\left(\frac{X_i}{\overline{X}}\right) && (31)\\
&\dot{\sim}\ \chi^2_{n-1} && (32)
\end{aligned}
$$

A Taylor approximation is

$$
\begin{aligned}
-2\ln(\Lambda) &= \frac{1}{\overline{X}} \sum \left(X_i - \overline{X}\right)^2 && (33)\\
&= (n-1)\frac{s^2}{\overline{X}} && (34)\\
&\dot{\sim}\ \chi^2_{n-1} && (35)
\end{aligned}
$$